

Our 2D metric is in "polar" type coordinates and looks circularly symmetric (only r -dep.).
 To add a third dimension in the simplest way, we should extend this 2D circular geometry into 3D using cylindrical coordinates $\{z, \rho, \psi\}$ on \mathbb{R}^3 .

$$\mathbb{R}^3 \text{ w/ } \{z, \rho, \psi\} \quad ds^2 = dz^2 + d\rho^2 + \rho^2 d\psi^2 \quad \begin{array}{l} z \in (-\infty, \infty) \\ \rho \in [0, \infty) \\ \psi \in [0, 2\pi) \end{array}$$

To embed we need: $z(r, \phi), \rho(r, \phi), \psi(r, \phi)$

Aligning axis and identifying angles we expect: $z(r), \rho(r), \psi = \phi$

To get $z(r), \rho(r)$ consider: $ds^2 = dz^2 + d\rho^2 + \rho^2 d\psi^2 = \left(\frac{dz}{dr}\right)^2 dr^2 + \left(\frac{d\rho}{dr}\right)^2 dr^2 + \rho^2 d\phi^2$

If $\rho^2 = r^2 + b^2$ and $\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2 = 1$ then $ds^2 = dr^2 + (r^2 + b^2) d\phi^2$

$$\Downarrow \quad \Downarrow$$

$$\frac{d\rho}{dr} = \frac{r}{\sqrt{r^2 + b^2}} \quad \left(\frac{dz}{dr}\right)^2 + \frac{r^2}{r^2 + b^2} = 1$$

$$\Downarrow$$

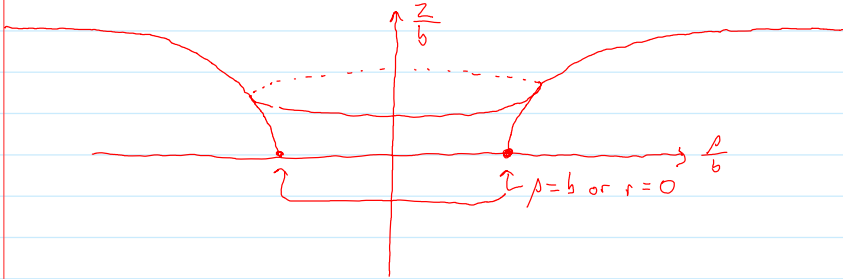
$$z(r) = b \sinh^{-1}\left(\frac{r}{b}\right) \text{ having chosen } z(0) = 0$$

$$\text{Note: for } r > 0 \Rightarrow z > 0 \quad \left[r = b \sinh\left(\frac{z}{b}\right) = \frac{b}{2} \left(e^{\frac{z}{b}} - e^{-\frac{z}{b}} \right) \right]$$

In terms of ρ we have:

$$z(\rho) = b \sinh^{-1}\left(\sqrt{\frac{\rho^2}{b^2} - 1}\right)$$

We can now plot $z(\rho)$ and revolve through ϕ around z to visualize the geometry:



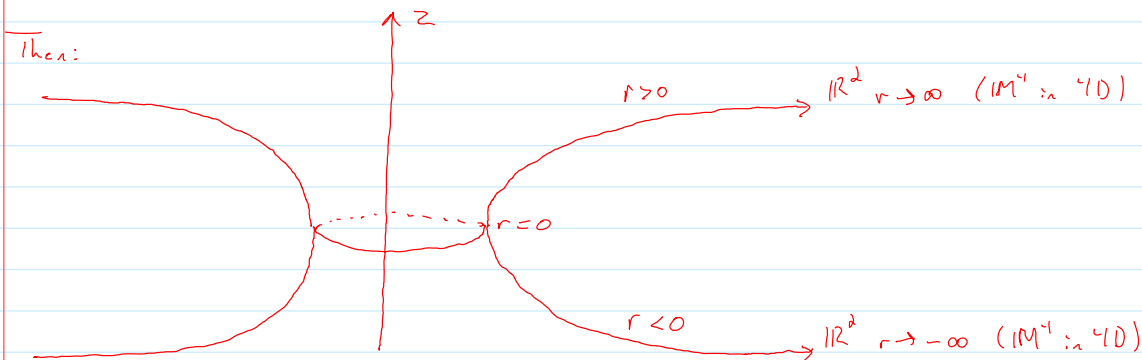
$$z(\rho) = b \sinh^{-1} \left(\sqrt{\frac{\rho^2}{b^2} - 1} \right)$$

$$r = \sqrt{\rho^2 - b^2}$$

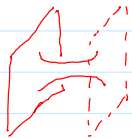
Recall that to restore Θ , we let each $s^1 \Rightarrow s^2$
 Then finally we can restore t to get the full 4D geometry.

This space is geodesically incomplete. This means if we pick a point and an initial direction, then for some choices (not all) the solutions to the geodesic equation for this geometry will terminate after a finite path length (others are ∞ or periodic). These include any path that moves towards $r=0$. Note this doesn't happen in flat space in polar coordinates since paths can move through $r=0$ continuously.

We can "maximally extend" this geometry to make it geodesically complete by allowing r to run negative, i.e. $r \in (-\infty, \infty)$ instead of $r \in [0, \infty)$. From $z(r) = b \sinh^{-1} \left(\frac{r}{b} \right)$ or $r = b \sinh \left(\frac{z}{b} \right)$ we see that $r < 0$ corresponds to $z < 0$.



This 2D space, and by extension its 4D version, now connects two distinct asymptotic geometries that share a common time. This is a wormhole geometry but not exactly of the type commonly seen in sci-fiction:



This case the wormhole connects different geometries (doorway)



More complicated case wormhole connects different parts of one geometry (shortcut)

These are smooth geometries (no ∞ curvature) so we might wonder if they can exist. Unfortunately, to solve $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and get these geometries requires $\rho < 0$ (negative energy density) which only comes from vacuum energy. The universe does have vacuum energy but it is uniform on large scales, and these solutions are obviously not. We do get non-uniform vacuum energy from quantum fluctuations, but good luck getting through a quantum sized wormhole!

But don't get too disappointed...

Consider the Schwarzschild geometry (in Schwarzschild coordinates):

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad \text{for now w/ } r > 2GM$$

To explore the geodesic completeness in this case we will use yet a new set of coordinates.

$$\left. \begin{aligned} T_{\pm} &\equiv \left(\frac{r}{2GM} - 1\right)^{1/2} e^{\frac{t}{4GM}} \sinh\left(\frac{t}{4GM}\right) \\ R_{\pm} &\equiv \left(\frac{r}{2GM} - 1\right)^{1/2} e^{\frac{t}{4GM}} \cosh\left(\frac{t}{4GM}\right) \end{aligned} \right\} \text{Kruskal coordinates}$$

↳ T_{\pm} to be explained later

Since $t \in (-\infty, \infty)$ and $r > 2GM$ we have $T_{\pm} \in (-\infty, \infty)$, $R_{\pm} \in (0, \infty)$

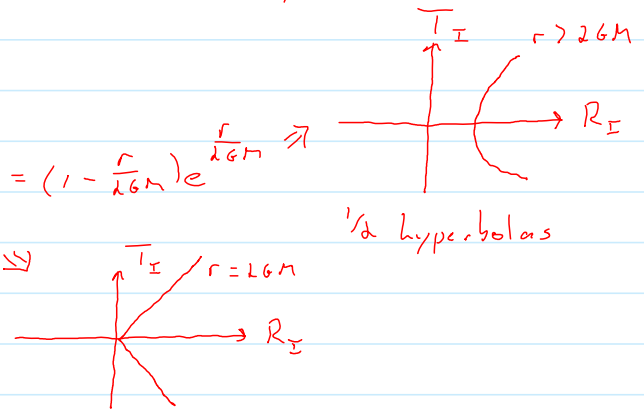
In terms of these: $ds^2 = \frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} (-dT_{\pm}^2 + dR_{\pm}^2) + r^2 d\Omega^2$

We will leave in some r -dependence because it will help identify points in the original description.
 ↳ Schwarzschild r

3 important aspects of Kruskal coordinates:

- Lines of constant r satisfy $T_{\pm} - R_{\pm} = \left(1 - \frac{r}{2GM}\right) e^{\frac{t}{4GM}}$

In particular for $r = 2GM$, $T_{\pm} = \pm R_{\pm}$



↳ hyperbolas

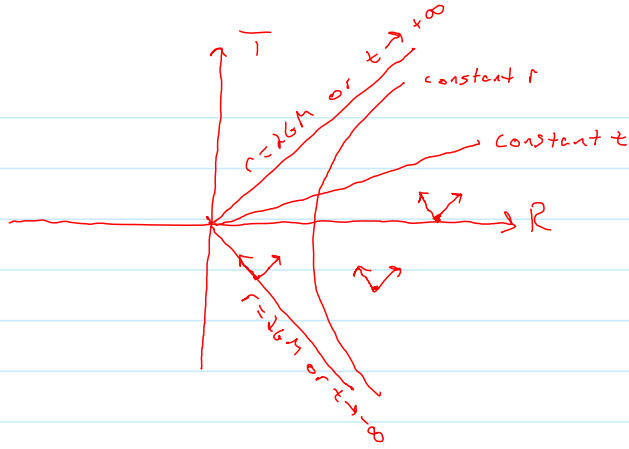
- Lines of constant t satisfy $T_{\pm} = R_{\pm} \tanh\left(\frac{t}{4GM}\right) \Rightarrow \frac{dT_{\pm}}{dR_{\pm}} = \tanh\left(\frac{t}{4GM}\right) = \text{constant}$

In particular $t \rightarrow \pm\infty \Rightarrow T_{\pm} \rightarrow \pm R_{\pm}$

- Light cones open at 45° everywhere on a T_{\pm} vs. R_{\pm} plot since $ds^2 = 0 = (-dT_{\pm}^2 + dR_{\pm}^2)$

$$\Downarrow \\ \frac{dT_{\pm}}{dR_{\pm}} = \pm 1$$

Putting this all together:



Now we know that a geodesic can start at $r > 2GM$ and go below $r = 2GM$, so we can extend this to $r < 2GM$ using:

$$T_{II} \equiv \left(1 - \frac{r}{2GM}\right)^{1/2} e^{\frac{t}{4GM}} \cosh\left(\frac{t}{4GM}\right)$$

$$R_{II} \equiv \left(1 - \frac{r}{2GM}\right)^{1/2} e^{\frac{t}{4GM}} \sinh\left(\frac{t}{4GM}\right)$$

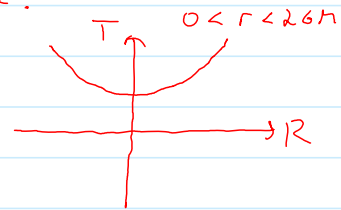
For $t \in (-\infty, \infty)$ and $r \in (0, 2GM)$ we have $T_{II} \in (0, \infty)$ and $R_{II} \in (-\infty, \infty)$

Note difference w/ T_I, R_I

In terms of these:

- $ds^2 = \frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} (-dT_{II}^2 + dR_{II}^2) + r^2 d\Omega^2$ same!

- Constant $r \Rightarrow T_{II}^2 - R_{II}^2 = \left(1 - \frac{r}{2GM}\right) e^{\frac{t}{2GM}} \Rightarrow$
 In particular for $r=0$ $T_{II}^2 - R_{II}^2 = 1$ singularity

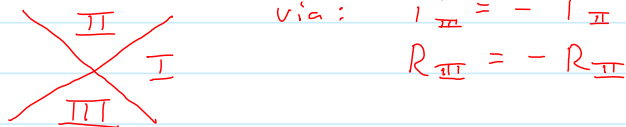


- Constant $t \Rightarrow T_{II} = R_{II} \coth\left(\frac{t}{4GM}\right) \Rightarrow T_{II} \rightarrow \pm R_{II}$ for $t \rightarrow \pm\infty$
- Since the metric is unchanged light-cones still open at 45° .

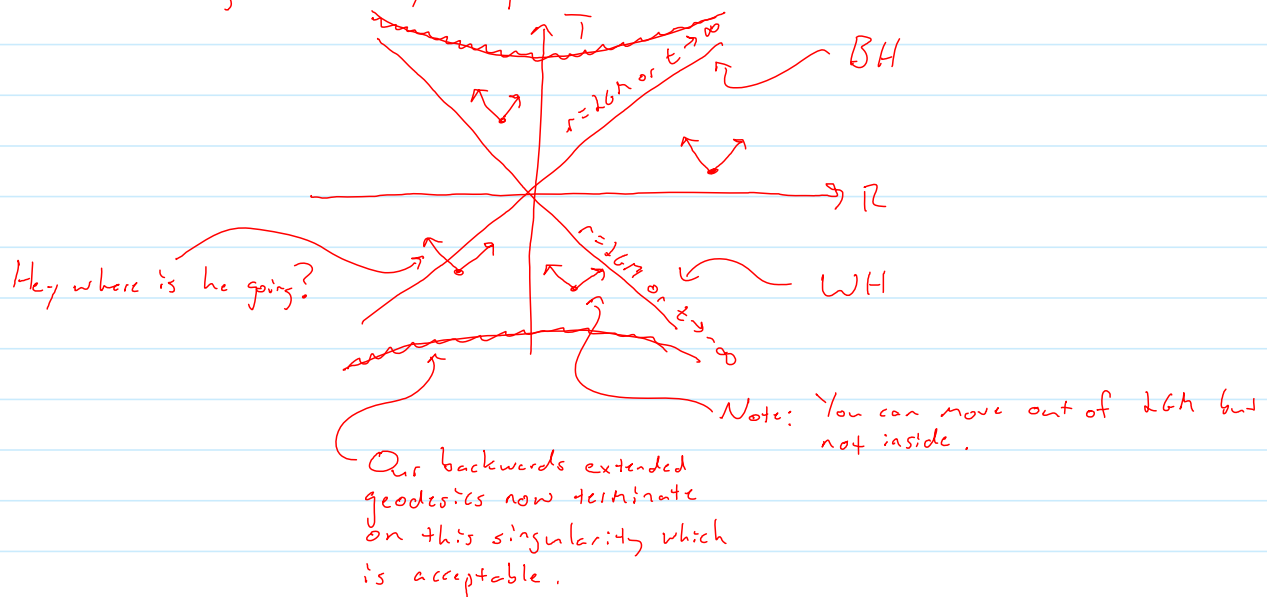
From our Kruskal diagram we immediately see that there are geodesics which, when traced backwards in time, terminate in a finite proper time, i.e. those which seem to come "out of" $r=2GM$ in the past.

We also see geodesics which terminate at finite proper time in the future since they hit the singularity at $r=0$. There isn't anything we can or should do about these. In fact, in more general contexts, the point(s) of termination of geodesics is used to define singularities.

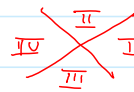
For the former type of geodesic, we can extend the geometry into a new region I will call III



With these choices it is obvious that ds^2 is the same and light-cones still open at 45° . Moreover we now see a singularity in the past and a white hole out of which things can only escape and never enter.

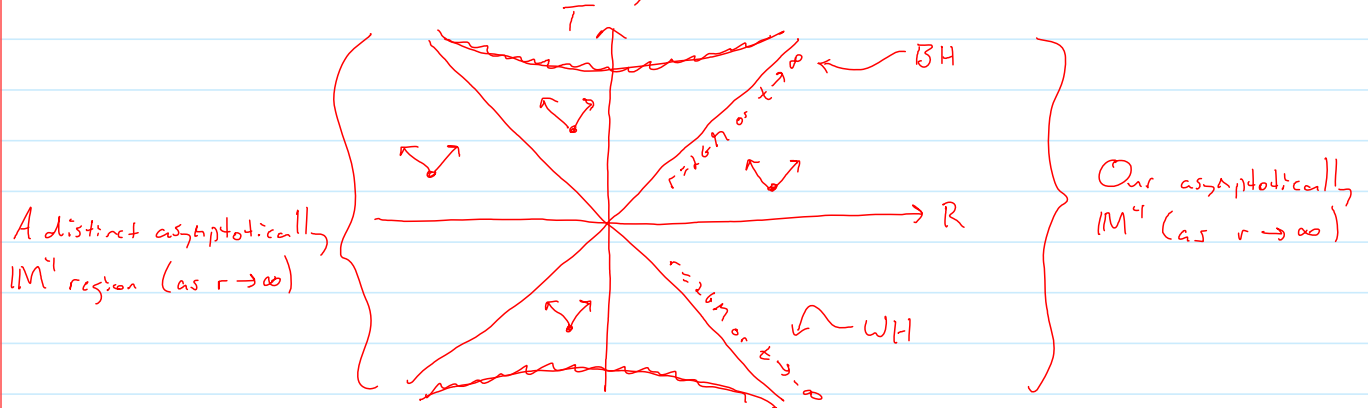


But now we see that paths leaving the WH also have yet another place to go, so we complete the story w/ region IV



via: $T_{IV} = -T_I$
 $R_{IV} = -R_I$

The complete picture of the "Maximally Extended Schwarzschild Geometry" is:



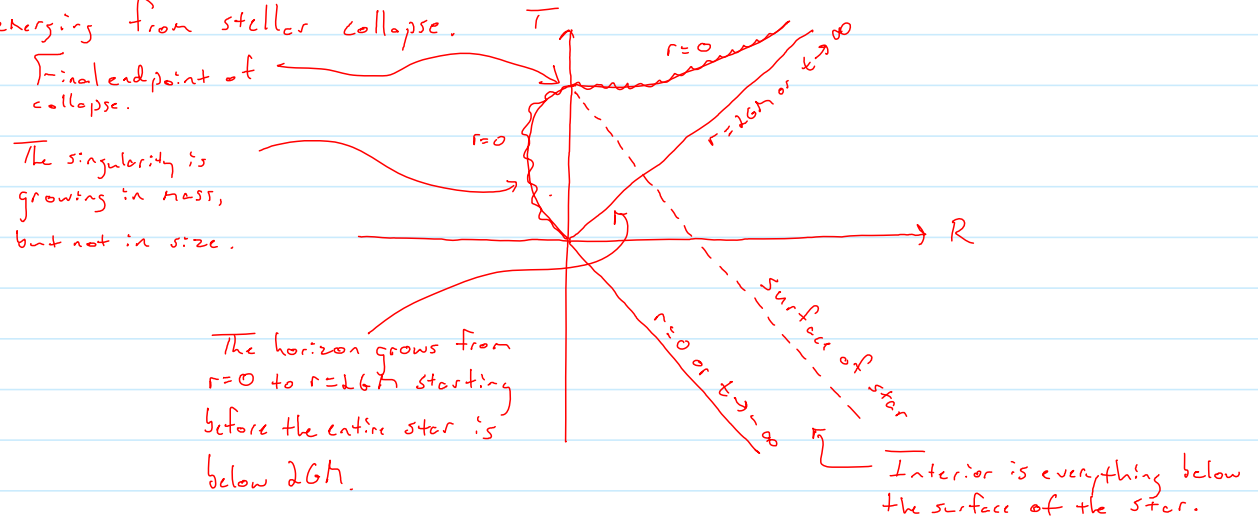
This is yet another example of a wormhole geometry, i.e. path connecting distinct asymptotic regions. However this has important differences. This wormhole is non-traversable since getting "in between" the two regions means ending up at $r=0$. Of course coming out of the WH, you get a choice of where to live, but it is a most irrevocable decision. Note that you and your twin could be born, part ways from the WH, inspect the two exteriors and finally meet and compare notes back inside of the BH. But you would only have $G_{H\pi}$ proper time to do so!

So is this really what a BH geometry should look like? Note our picture relied on $t \in (-\infty, \infty)$.

2 problems w/ this idea of an "eternal" BH:

- The universe has a finite age
- Astrophysical BHs are "born" from stellar collapse

In fact we can amend the Kruskal diagram for the more realistic scenario of BHs emerging from stellar collapse.



Once the surface hits $r=0$, we should just see the usual BH geometry. Before that, the details will be governed by the particulars of the interior solution.